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Generating functional for strong and nonleptonic weak interactions*

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Abstract: The generating functional for Green functions of quark currents is given in closed form to next-to-leading order in the low-energy expansion for chiral SU(3), including one-loop amplitudes with up to three meson propagators. Matrix elements and form factors for strong and nonleptonic weak processes with at most six external states can be extracted from this functional by performing three-dimensional flavour traces. To implement this procedure, a Mathematica© program is provided that evaluates amplitudes with at most six external mesons, photons (real or virtual) and virtual $W^{\pm}$ (semileptonic form factors). The program is illustrated with several examples that can be compared with existing calculations.

Keywords: Spontaneous Symmetry Breaking, Chiral Lagrangians, Kaon Physics.

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Contents

1. Introduction

Chiral perturbation theory [1−4] is the effective field theory of the standard model at low energies. In the mesonic sector, the state of the art is next-to-next-to-leading order in the low-energy expansion for strong processes [5] and next-to-leading order for nonleptonic weak transitions.

Although most one-loop amplitudes for processes of physical interest have already been calculated a compact closed form for such amplitudes would still be useful, both in principle and in practice. The generating functional for Green functions of quark currents is the relevant object for this purpose. The generating functional to next-to-leading order was already calculated in the classic papers of Gasser and Leutwyler [2, 3] for the strong interactions, including electromagnetic and semileptonic weak form factors, and by Kambor, Missimer and Wyler [6] for nonleptonic weak processes. In both cases, the calculation was limited to processes where at most two meson propagators occur in one-loop amplitudes.

More recently, the generating functional for chiral SU(2) was obtained in closed form for transitions involving up to three propagators in one-loop amplitudes: at most three pions in ref. [7] and at most two pions and a photon in ref. [8]. The purpose of this article is to calculate the analogous generating functional with up to three meson propagators for chiral SU(3) for both strong and nonleptonic weak transitions. The corresponding SU(3) calculation with dynamical photons is in preparation [9].

Transitions involving at most three meson propagators in one-loop amplitudes comprise almost all processes of physical interest at next-to-leading order. More precisely, the complete strong and nonleptonic weak amplitudes and form factors to that order can be extracted from the generating functional for transitions of at most $O(\phi^6)$, i.e. with at most six external mesons, photons (counting as $O(\phi^2)$) and virtual $W^\pm$. Although most of such processes have already been calculated to next-to-leading order the generating functional provides a closed expression for such amplitudes with the same conventions and notation. As experience has shown, such a presentation greatly facilitates comparison with previous work and eventual detection of misprints and other errors. In addition to reproducing previous results in a simple and straightforward manner, several nonleptonic transitions have been calculated for the first time [10] with the generating functional presented here.

Although the extraction of amplitudes or form factors from the generating functional boils down to performing three-dimensional flavour traces such a procedure may turn out to be quite time consuming, especially for transitions with three-propagator loop contributions. To facilitate this work, we provide the Mathematica© [11] program Ampcalculator, written by one of us (R.U.), that evaluates the necessary traces upon input of at most six external states (mesons, photons, virtual $W^\pm$) with their corresponding momenta. Use of the program is straightforward and does not require detailed knowledge of Mathematica©.

The generating functional is introduced in section 2 and its low-energy expansion is discussed. The generating functional of next-to-leading order is treated in section 3 for the strong interactions. We extend the work of Gasser and Leutwyler [2, 3] by calculating the explicit form of the one-loop functional for up to three meson propagators and to at most $O(\phi^6)$. The renormalization of the generating functional is briefly reviewed. In section 4 the nonleptonic weak interactions are included. The additional terms in the generating functional linear in the weak coupling constants G_8, G_{27} are calculated under the same conditions as for the strong part. In section 5 we discuss the Mathematica© program Ampcalculator for evaluating amplitudes and form factors for a given set of external states to next-to-leading order. The program is exemplified for the decays $K^+ \to \pi^0 l^+ \nu_l$ (K^+_{l3}), $K^- \to \pi^- \pi^0$, $K^0 \to \pi^0 \gamma\gamma$ and the results are compared with those in the literature. Our conclusions are summarized in section 6. Details about the input for the program Ampcalculator, one-loop integrals and constituent functions and the strong and weak lagrangians of $O(p^4)$ are collected in three appendices.

2. Generating functional of Green functions

The generating functional for Green functions of quark currents is defined in terms of the vacuum transition amplitude in the presence of external fields,

$$e^{iZ[v,a,s,p]} = < 0 \text{ out} | 0 \text{ in} >_{v,a,s,p} , \tag{2.1}$$

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associated with the lagrangian [2, 3]

$$\mathcal{L} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - ip\gamma_5)q\,, \tag{2.2}$$

where $\mathcal{L}^0_{\rm QCD}$ is the QCD lagrangian with three massless quarks and v_μ, a_μ, s, p are three-dimensional hermitian matrix fields.

Green functions of quark currents are obtained by functional differentiation of $Z[v,a,s,p]$ in the usual way. To implement explicit chiral symmetry breaking, the external scalar field s is set equal to the quark mass matrix $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$ at the end of the calculation. The external spin-1 fields generate electromagnetic and semileptonic form factors with the assignments

$$\begin{aligned} r^\mu &\doteq v^\mu + a^\mu = -e\,QA^\mu\,, \\ l^\mu &\doteq v^\mu - a^\mu = -e\,QA^\mu - \frac{e}{\sqrt{2}\sin\theta_W}(W^{\mu^+}T_+ + \text{h.c.}) \end{aligned} \tag{2.3}$$

for external photons (A_μ) and W bosons ($W^\pm_\mu$). θ_W is the weak mixing angle, Q is the quark charge matrix and T_+ contains the relevant elements of the Cabibbo-Kobayashi-Maskawa matrix:

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

At the hadronic level, the generating functional may be calculated in terms of an effective lagrangian of pseudoscalar mesons and external fields [4],

$$e^{iZ[v,a,s,p]} = \int [dU(\varphi)] e^{i\int d^4x\,\mathcal{L}_{\rm eff}\,(U,v,a,s,p)}\,. \tag{2.5}$$

The mesonic effective chiral lagrangian will be needed to next-to-leading order here, including both strong and nonleptonic weak parts:

$$\mathcal{L}_{\rm eff}\,(U,v,a,s,p) = \mathcal{L}_2 + \mathcal{L}_4 + \cdots\,. \tag{2.6}$$

The strong chiral lagrangian takes the well-known form [3]

$$\mathcal{L}^S_2 = \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\rangle\,, \qquad \chi = 2B(s + ip) \tag{2.7}$$

in terms of the pion decay constant and the quark condensate in the chiral limit:

$$\begin{aligned} F_\pi &= F[1 + O(m_q)] = 92.4\,\mathrm{MeV} \\ \langle 0|\bar{u}u|0\rangle &= -F^2B[1 + O(m_q)]\,. \end{aligned} \tag{2.8}$$

$\langle\ldots\rangle$ denotes the 3-dimensional flavour trace. The strong chiral lagrangian $\mathcal{L}^S_4$ of $O(p^4)$ [3] is reproduced in appendix C and the nonleptonic weak lagrangian will be discussed in section 4.

The chiral expansion of $\mathcal{L}_{\rm eff}$ induces a corresponding expansion for the generating functional:

$$Z = Z_2 + Z_4 + \cdots . \tag{2.9}$$

At lowest order, the functional Z_2 is equal to the classical action

$$Z_2[v,a,s,p] = \int d^4x\, \mathcal{L}_2(\bar{U}, v, a, s, p) \tag{2.10}$$

where $\bar{U} = \bar{U}[v,a,s,p]$ is to be understood as a functional of the external fields via the equation of motion (EOM) for $\mathcal{L}_2$ (given here for the strong interactions only):

$$\Box U U^\dagger - U \Box U^\dagger = \chi U^\dagger - U \chi^\dagger - \frac{1}{3}\langle \chi U^\dagger - U\chi^\dagger \rangle \mathbb{1} . \tag{2.11}$$

Mesonic amplitudes at lowest order can therefore be read off directly from the lowest-order lagrangian (2.7) and the corresponding nonleptonic weak lagrangian $\mathcal{L}_2^W$ in (4.1) by using an explicit parametrization of the matrix-valued meson field $U(\varphi)$, e.g., the exponential parametrization

$$U(\varphi) = \exp\left(\frac{i\lambda_a\varphi^a}{F}\right), \qquad \frac{1}{\sqrt{2}}\lambda_a\varphi^a = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix} . \tag{2.12}$$

The matrix field (2.12) also defines our sign conventions for mesonic amplitudes generated by the Mathematica© program Ampcalculator to be discussed in section 5.

3. Generating functional of $\mathbf{O(p^4)}$

The generating functional at next-to-leading order consists of three parts:

$$Z_4 = Z_4^{\rm tree} + Z_4^{L=1} + Z_{\rm WZW} . \tag{3.1}$$

The tree-level part $Z_4^{\rm tree}$ is given by the action for the next-to-leading lagrangian $\mathcal{L}_4(U, v, a, s, p)$ to be taken again at the classical solution $\bar{U}$ satisfying the EOM (2.11). The Wess-Zumino-Witten functional $Z_{\rm WZW}$ [12] accounts for the chiral anomaly. We will not reproduce its explicit form here but it will be implemented in the Mathematica© program Ampcalculator (cf. section 5).

The one-loop functional $Z_4^{L=1}$ is calculated in the standard way. We repeat the main steps here in order to define our notation. The matrix field $U(\varphi)$ is expanded around the classical solution $\bar{U} = \bar{u}^2$ in terms of a traceless, hermitian fluctuation matrix ξ:

$$U = u^2 = \bar{u}\left(\mathbb{1} + i\xi - \frac{1}{2}\xi^2 + \cdots\right)\bar{u} . \tag{3.2}$$

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Working with dimensional regularization in d dimensions, one obtains

$$e^{iZ_4^{L=1}} = \int d\mu[\xi] \, e^{-\frac{i}{2} F^2 \int d^d x \, \xi^a(x) \mathcal{D}^{ab}(x) \xi^b(x)} , \qquad \xi = \frac{1}{\sqrt{2}} \lambda_a \xi^a . \tag{3.3}$$

The measure $d\mu[\xi]$ is defined in such a way that $Z_4^{L=1}$ vanishes when the external fields v, a and p are set to zero and s is set equal to the quark mass matrix. In the same limit, the differential operator $\mathcal{D}^{ab}$ turns into the Klein-Gordon operator $\mathcal{D}_0^{ab}$ with the appropriate meson masses to leading order in the chiral expansion.[1] The one-loop functional can then be written in the form

$$Z_4^{L=1} = \frac{i}{2} \ln \mathrm{Det} \, \frac{\mathcal{D}}{\mathcal{D}_0} = \frac{i}{2} \, \mathrm{Tr} \ln \frac{\mathcal{D}}{\mathcal{D}_0} . \tag{3.4}$$

Splitting the differential operator $\mathcal{D}$ into the Klein-Gordon operator $\mathcal{D}_0$ and a remainder δ, one gets

$$Z_4^{L=1} = \frac{i}{2} \, \mathrm{Tr} \ln \frac{\mathcal{D}_0 + \delta}{\mathcal{D}_0} = \frac{i}{2} \, \mathrm{Tr} \ln \left(\mathbb{1} + \delta \mathcal{D}_0^{-1} \right) = \frac{i}{2} \, \mathrm{Tr} \ln (\mathbb{1} - \delta \Delta) . \tag{3.5}$$

The diagonal matrix Δ is the inverse of $-\mathcal{D}_0$ and it contains the Feynman propagators of pseudoscalar mesons:

$$\Delta_{ab}(x-y) = \delta_{ab} \int \frac{d^d p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - M_a^2 + i\epsilon} . \tag{3.6}$$

For the strong interactions δ is of the form [3]

$$\delta_S = \left\{ \hat{\Gamma}^\mu , \partial_\mu \right\} + \hat{\Gamma}^\mu \hat{\Gamma}_\mu + \bar{\sigma} \tag{3.7}$$

with

$$\begin{aligned}
\hat{\Gamma}^\mu_{ab} &= -\frac{1}{2} \langle [\lambda_a, \lambda_b] \Gamma^\mu \rangle , \\
\bar{\sigma}_{ab} &= \hat{\sigma}_{ab} - M_a^2 \delta_{ab} , \\
\hat{\sigma}_{ab} &= \frac{1}{2} \langle [\lambda_a, y_\mu][\lambda_b, y^\mu] \rangle + \frac{1}{4} \langle \{\lambda_a, \lambda_b\} \sigma \rangle , \\
y_\mu &= \frac{1}{2} \bar{u}^\dagger D_\mu \bar{U} \bar{u}^\dagger = -\frac{1}{2} \bar{u} D_\mu \bar{U}^\dagger \bar{u} , \\
\Gamma_\mu &= \frac{1}{2} [\bar{u}^\dagger, \partial_\mu \bar{u}] - \frac{1}{2} i \bar{u}^\dagger r_\mu \bar{u} - \frac{1}{2} i \bar{u} l_\mu \bar{u}^\dagger , \\
\sigma &= \frac{1}{2} (\bar{u} \chi^\dagger \bar{u} + \bar{u}^\dagger \chi \bar{u}^\dagger) .
\end{aligned} \tag{3.8}$$

To include one-loop contributions with up to three meson propagators, the one-loop functional (3.5) must be expanded to third order in δ:

$$Z_4^{L=1} = -\frac{i}{2} \, \mathrm{Tr}(\delta\Delta) - \frac{i}{4} \, \mathrm{Tr}(\delta\Delta\delta\Delta) - \frac{i}{6} \, \mathrm{Tr}(\delta\Delta\delta\Delta\delta\Delta) + \cdots . \tag{3.9}$$

[1] Since we work in the isospin limit throughout this paper the differential operator $\mathcal{D}_0$ is diagonal.

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The functional (3.9) suffices for transitions up to $O(\phi^6)$, i.e. with at most six external mesons where an external photon counts as $O(\phi^2)$. We will therefore limit the further discussion to processes of at most $O(\phi^6)$. Since the external fields v_μ and s couple to at least two pseudoscalar meson fields whereas a_μ and p couple to at least one field, the functional needs to be expanded to third order in v_μ and s and to sixth order in a_μ and p. Keeping in mind that the matrices $\hat{\Gamma}^\mu$ and $\bar{\sigma}$ are of $O(\phi^2)$, the strong one-loop functional assumes the form

$$
\begin{aligned}
Z_4^{L=1} = &-\frac{i}{2}\,\mathrm{Tr}\left(\bar{\sigma}(x)\Delta(0) + \hat{\Gamma}^\mu(x)\hat{\Gamma}_\mu(x)\Delta(0)\right) - \\
&-\frac{i}{4}\,\mathrm{Tr}\Big(\{\hat{\Gamma}^\mu(x), \partial_{x_\mu}\}\Delta(x-y)\{\hat{\Gamma}^\nu(y), \partial_{y_\nu}\}\Delta(y-x) + \\
&\qquad + 2\{\hat{\Gamma}^\mu(x), \partial_{x_\mu}\}\Delta(x-y)\hat{\Gamma}^\nu(y)\hat{\Gamma}_\nu(y)\Delta(y-x) + \\
&\qquad + 2\{\hat{\Gamma}^\mu(x), \partial_{x_\mu}\}\Delta(x-y)\bar{\sigma}(y)\Delta(y-x) + \\
&\qquad + 2\hat{\Gamma}^\mu(x)\hat{\Gamma}_\mu(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-x) + \bar{\sigma}(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-x)\Big) - \\
&-\frac{i}{6}\,\mathrm{Tr}\Big(\{\hat{\Gamma}^\mu(x), \partial_{x_\mu}\}\Delta(x-y)\{\hat{\Gamma}^\nu(y), \partial_{y_\nu}\}\Delta(y-z)\{\hat{\Gamma}^\lambda(z), \partial_{z_\lambda}\}\Delta(z-x) + \\
&\qquad + 3\{\hat{\Gamma}^\mu(x), \partial_{x_\mu}\}\Delta(x-y)\{\hat{\Gamma}^\nu(y), \partial_{y_\nu}\}\Delta(y-z)\bar{\sigma}(z)\Delta(z-x) + \\
&\qquad + 3\bar{\sigma}(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-z)\{\hat{\Gamma}^\mu(z), \partial_{z_\mu}\}\Delta(z-x) + \\
&\qquad + \bar{\sigma}(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-z)\bar{\sigma}(z)\Delta(z-x)\Big) + \cdots \\
= &-\frac{i}{2}\,\mathrm{Tr}\left(\bar{\sigma}(x)\Delta(0) + \hat{\Gamma}^\mu(x)\hat{\Gamma}_\mu(x)\Delta(0)\right) - \\
&-\frac{i}{4}\,\mathrm{Tr}\Big(\hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\Delta(y-x) + \\
&\qquad + \hat{\Gamma}^\mu(x)\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{x_\mu}\Delta(y-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\partial_{x_\mu}\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\Delta(y-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{x_\mu}\partial_{y_\nu}\Delta(y-x) + \\
&\qquad + 2\hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\hat{\Gamma}^\nu(y)\hat{\Gamma}_\nu(y)\Delta(y-x) - \\
&\qquad - 2\hat{\Gamma}^\mu(x)\Delta(x-y)\hat{\Gamma}^\nu(y)\hat{\Gamma}_\nu(y)\partial_{x_\mu}\Delta(y-x) + \\
&\qquad + 2\hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\bar{\sigma}(y)\Delta(y-x) - 2\hat{\Gamma}^\mu(x)\Delta(x-y)\bar{\sigma}(y)\partial_{x_\mu}\Delta(y-x) + \\
&\qquad + 2\hat{\Gamma}^\mu(x)\hat{\Gamma}_\mu(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-x) + \bar{\sigma}(x)\Delta(x-y)\bar{\sigma}(y)\Delta(y-z)\Big) - \\
&-\frac{i}{6}\,\mathrm{Tr}\Big(\hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{z_\lambda}\Delta(z-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{z_\lambda}\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{x_\mu}\Delta(z-x) + \\
&\qquad + \hat{\Gamma}^\mu(x)\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{x_\mu}\partial_{z_\lambda}\Delta(z-x) + \\
&\qquad + \hat{\Gamma}^\mu(x)\partial_{x_\mu}\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{z_\lambda}\Delta(y-z)\hat{\Gamma}^\lambda(z)\Delta(z-x) + \\
&\qquad + \hat{\Gamma}^\mu(x)\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\partial_{z_\lambda}\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{x_\mu}\Delta(z-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{x_\mu}\partial_{z_\lambda}\Delta(z-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\partial_{x_\mu}\partial_{y_\nu}\Delta(x-y)\hat{\Gamma}^\nu(y)\Delta(y-z)\hat{\Gamma}^\lambda(z)\partial_{z_\lambda}\Delta(z-x) - \\
&\qquad - \hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\partial_{z_\lambda}\Delta(y-z)\hat{\Gamma}^\lambda(z)\Delta(z-x) + \\
&\qquad + 3\hat{\Gamma}^\mu(x)\partial_{x_\mu}\Delta(x-y)\hat{\Gamma}^\nu(y)\partial_{y_\nu}\Delta(y-z)\bar{\sigma}(z)\Delta(z-x) +
\end{aligned}
$$

$$
\begin{aligned}
&+3\hat\Gamma^\mu(x)\partial_{y_\nu}\Delta(x-y)\hat\Gamma^\nu(y)\Delta(y-z)\bar\sigma(z)\partial_{x_\mu}\Delta(z-x)-\\
&-3\hat\Gamma^\mu(x)\Delta(x-y)\hat\Gamma^\nu(y)\partial_{y_\nu}\Delta(y-z)\bar\sigma(z)\partial_{x_\mu}\Delta(z-x)-\\
&-3\hat\Gamma^\mu(x)\partial_{x_\mu}\partial_{y_\nu}\Delta(x-y)\hat\Gamma^\nu(y)\Delta(y-z)\bar\sigma(z)\Delta(z-x)+\\
&+3\bar\sigma(x)\Delta(x-y)\bar\sigma(y)\Delta(y-z)\hat\Gamma^\mu(z)\partial_{z_\mu}\Delta(z-x)-\\
&-3\bar\sigma(x)\Delta(x-y)\bar\sigma(y)\partial_{z_\mu}\Delta(y-z)\hat\Gamma^\mu(z)\Delta(z-x)+\\
&+\bar\sigma(x)\Delta(x-y)\bar\sigma(y)\Delta(y-z)\bar\sigma(z)\Delta(z-x)\Big)+\cdots. \qquad (3.10)
\end{aligned}
$$

Integration by parts was used in eq. (3.10) to shift the derivatives from the matrices containing the fields to the propagators. In terms of the functions A and G_i defined in appendix B, the strong one-loop functional to $O(\phi^6)$ can be written

$$
\begin{aligned}
Z_4^{L=1}[\bar U,a,v,s,p]=&\int d^4x\left(\frac{1}{2}\sum_P A(M_P^2)\bar\sigma_{PP}(x)+\frac{1}{4}\sum_{P,Q}(A(M_P^2)+A(M_Q^2))\hat\Gamma_{\rho PQ}(x)\hat\Gamma^\rho{}_{QP}(x)\right)+\\
&+\int d^4x d^4y\frac{d^4p}{(2\pi)^4}\mathrm{e}^{-ip(x-y)}\sum_{P,Q,R}\Big(G^{\mu\nu}_{1\,PQ}(p)\hat\Gamma_{\mu QP}(x)\hat\Gamma_{\nu PQ}(y)+\\
&\qquad+G^{\mu}_{2\,PQ}(p)\hat\Gamma_{\mu QP}(x)\bar\sigma_{PQ}(y)+\\
&\qquad+G_{3\,PQ}(p)\big(\bar\sigma_{QP}(x)\bar\sigma_{PQ}(y)+\\
&\qquad+2\hat\Gamma_{\rho QR}(x)\hat\Gamma^\rho{}_{RP}(x)\bar\sigma_{PQ}(y)\big)\Big)+\\
&+\int d^4x d^4y d^4z\frac{d^4p_a}{(2\pi)^4}\frac{d^4p_b}{(2\pi)^4}\mathrm{e}^{ip_a(z-x)}\mathrm{e}^{ip_b(y-z)}\times\\
&\times\sum_{P,Q,R}\Big(G^{\mu\nu\lambda}_{4\,PQR}(p_a,p_b)\hat\Gamma_{\mu QP}(x)\hat\Gamma_{\nu PR}(y)\hat\Gamma_{\lambda RQ}(z)+\\
&\qquad+G^{\mu\nu}_{5\,PQR}(p_a,p_b)\hat\Gamma_{\mu QP}(x)\hat\Gamma_{\nu PR}(y)\bar\sigma_{RQ}(z)+\\
&\qquad+G^{\mu}_{6\,PQR}(p_a,p_b)\bar\sigma_{QP}(x)\bar\sigma_{PR}(y)\hat\Gamma_{\mu RQ}(z)+\\
&\qquad+G_{7\,PQR}(p_a,p_b)\bar\sigma_{QP}(x)\bar\sigma_{PR}(y)\bar\sigma_{RQ}(z)\Big). \qquad (3.11)
\end{aligned}
$$

The divergent parts of the one-loop functional (3.11) are contained in the functions $A, B(0)$ defined in eqs. (B.1), (B.14). Renormalization amounts to absorbing those divergences in the low-energy constants L_i of the next-to-leading lagrangian (C.1), rendering those constants scale dependent at the same time. In other words, to obtain the renormalized, scale-independent generating functional Z_4 in eq. (3.1), the divergences $\Lambda(\mu)$ defined in (B.12) are to be dropped in $Z_4^{L=1}$, with the L_i being replaced by the renormalized constants $L_i^r(\mu)$ in the local functional Z_4^{tree}. The analogous procedure will be understood for the nonleptonic weak part of the generating functional of $O(p^4)$ discussed in the following section.

The momentum flow shown in figure 1 was chosen to fulfill the following criteria:

- The generating functional should contain as few terms as possible.
- The numerator of the three-point functions, i.e. $(k^2-M^2)((k-p_a)^2-M^2)((k-p_b)^2-M^2)$, should be symmetric under the interchange of p_a and p_b.

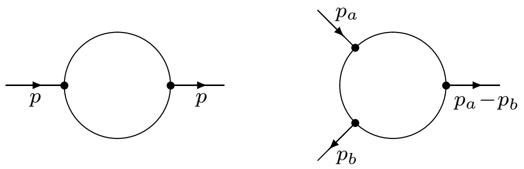

Figure 1: Definition of external momenta.

4. Nonleptonic weak interactions

To describe $|\Delta S| = 1$ nonleptonic weak processes, one has to insert the nonleptonic weak hamiltonian in the appropriate Green functions. At the hadronic level, this amounts to including exactly one vertex from the effective nonleptonic weak lagrangian in the various parts of the generating functional.

At leading order, the nonleptonic weak lagrangian of $O(G_F\, p^2)$ describing $|\Delta S| = 1$ processes is given by [13]

$$\begin{aligned}
\mathcal{L}_2^W &= F^4 \left[G_8 \langle \lambda L_\mu L^\mu \rangle + G_{27} \left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) \right] + \text{h.c.}\,, \\
\lambda &= \frac{(\lambda_6 - i\lambda_7)}{2}\,, \qquad L_\mu = i\, U^\dagger D_\mu U\,,
\end{aligned} \tag{4.1}$$

where $L_{\mu ij}$ denotes the ij-component of the matrix L_μ. The weak mass term proportional to $\langle \lambda (U^\dagger \chi + \chi^\dagger U) \rangle$ can be transformed away by a suitable field redefinition [14, 6]. The octet and 27-plet coupling constants G_8 and G_{27} can be extracted from $K \to \pi\pi$ decay rates [15]. We work consistently to first order in G_8 and G_{27}.

To $O(p^2)$, with terms linear in G_8, G_{27} included, the generating functional is still given by the classical action (2.10) with

$$\mathcal{L}_2 = \mathcal{L}_2^S + \mathcal{L}_2^W\,. \tag{4.2}$$

The classical solution $\bar{U}$ now satisfies a modified EOM where the weak part due to $\mathcal{L}_2^W$ is added to (2.11).

At $O(p^4)$, the structure of the generating functional is again given by eq. (3.1). The tree-level part Z_4^{tree} now contains the action for the nonleptonic weak lagrangian of $O(G_F p^4)$ reproduced in eqs. (C.4), (C.6). The chiral anomaly affects weak amplitudes both through the weak part in the classical solution $\bar{U}$ in Z_{WZW} and intrinsically through terms proportional to the ϵ-tensor in the weak lagrangian $\mathcal{L}_4^W$ in (C.3) [16].

It remains to determine the weak part of the one-loop functional $Z_4^{L=1}$. The operator δ defined in eq. (3.5) is now of the form

$$\delta = \delta_S + \delta_W\,. \tag{4.3}$$

The weak perturbation δ_W is given to first order in G_F by the following expression [6]:

$$\begin{aligned}
\delta_W &= \{\hat{\Gamma}_\mu^W, \partial^\mu\} + \{\hat{\Gamma}_\rho^W, \hat{\Gamma}^\rho\} + \hat{\sigma}^W\,, \\
\hat{\Gamma}_\mu^W &= -\frac{1}{2} N_\mu^- + \frac{1}{2} [T - T^\dagger, \hat{\Gamma}_\mu]\,,
\end{aligned}$$

$$
\begin{aligned}
\hat{\sigma}^W &= \hat{\omega} + \frac{1}{2}[d^\mu, N_\mu^+] - \frac{1}{2}[d^\mu,[d_\mu,\alpha]] - \frac{1}{2}\{\bar{\alpha},\hat{\sigma}\} + T\bar{\sigma} + \bar{\sigma}T^\dagger\,, \\
T_{ab} &= \frac{\hat{\sigma}^0_{ac}\alpha^0_{cb}}{M_b^2 - M_a^2}\,, \qquad d_\mu A = \partial_\mu A + [\hat{\Gamma}^\mu, A]\,, \qquad \bar{\alpha} = \alpha - \alpha^0\,.
\end{aligned} \tag{4.4}
$$

The matrices $\hat{\sigma}^0$ and α^0 are obtained from $\hat{\sigma}$ and α by switching off the external fields. In the weak perturbation there are no terms without external fields because a diagonalizing transformation similar to the one in ref. [17] has been performed. The components of the antisymmetric matrix N_μ^- are

$$
\begin{aligned}
N^-_{\mu ab} = &-\frac{1}{4}\kappa\langle\{K_{32},y_\mu\}[\lambda_a,\lambda_b]\rangle - \frac{1}{2}\kappa\langle K_{32}\,(\lambda_a y_\mu \lambda_b - \lambda_b\, y_\mu \lambda_a)\rangle - \zeta\langle[\lambda_a,\lambda_b]K\rangle\langle K y_\mu\rangle + \\
&+\frac{1}{2}\zeta\langle\lambda_a[K,y_\mu]\rangle\langle K\lambda_b\rangle - \frac{1}{2}\zeta\langle\lambda_b[K,y_\mu]\rangle\langle K\lambda_a\rangle + \text{h.c.}
\end{aligned} \tag{4.5}
$$

The coefficient κ stands for

$$
\kappa = 4\,G_8\,F^2 \tag{4.6}
$$

and the matrix K is defined by matrix elements

$$
K_{ij} = \bar{u}\lambda_{ij}\bar{u}^\dagger \qquad \text{with} \qquad (\lambda_{ij})_{kl} = \delta_{ik}\delta_{jl}\,. \tag{4.7}
$$

The operators $\zeta\langle K\mathcal{O}\rangle\langle K\mathcal{P}\rangle$ are due to the 27-plet contribution and are defined as

$$
\zeta\langle K\mathcal{O}\rangle\langle K\mathcal{P}\rangle = 4G_{27}F^2 t_{ij,kl}\langle K_{ij}\mathcal{O}\rangle\langle K_{kl}\mathcal{P}\rangle\,, \tag{4.8}
$$

with coefficients

$$
\begin{aligned}
t_{23,11} &= t_{11,23} = \frac{1}{2}\,, \qquad t_{21,13} = t_{13,21} = \frac{1}{3}\,, \\
t_{ij,kl} &= 0 \qquad\qquad\qquad \text{otherwise}\,.
\end{aligned} \tag{4.9}
$$

The expressions for the remaining symmetric matrices are

$$
N^+_{\mu ab} = \frac{1}{4}\kappa\langle[K_{32},y_\mu]\{\lambda_a,\lambda_b\}\rangle + \frac{1}{2}\zeta\langle\lambda_a[K,y_\mu]\rangle\langle K\lambda_b\rangle + \frac{1}{2}\zeta\langle\lambda_b[K,y_\mu]\rangle\langle K\lambda_a\rangle + \text{h.c.}\,, \tag{4.10}
$$

$$
\begin{aligned}
\hat{\omega}_{ab} = &-\frac{1}{4}\kappa\langle\{\lambda_a,\lambda_b\}\{y^2,K_{32}\}\rangle + \frac{1}{4}\kappa\langle K_{32}\left(\lambda_a y^2\lambda_b + \lambda_b y^2 \lambda_a\right)\rangle + \\
&+\frac{1}{4}\kappa\langle\{K_{32},y^\rho\}(\lambda_a\, y_\rho\lambda_b + \lambda_b\, y_\rho\lambda_a)\rangle - \frac{1}{4}\kappa\langle y^\rho K_{32} y_\rho\{\lambda_a,\lambda_b\}\rangle - \\
&-\frac{1}{2}\zeta\langle[K,y^\rho]\lambda_a\rangle\langle[K,y_\rho]\lambda_b\rangle - \frac{1}{2}\zeta\langle K y^\rho\rangle\langle\{K,y_\rho\}\{\lambda_a,\lambda_b\}\rangle + \\
&+\zeta\langle K\,(\lambda_a y^\rho\lambda_b + \lambda_b y^\rho\lambda_a)\rangle\langle K y_\rho\rangle + \text{h.c.}\,,
\end{aligned} \tag{4.11}
$$

$$
\alpha_{ab} = \frac{1}{4}\kappa\langle\{\lambda_a,\lambda_b\}K_{32}\rangle + \frac{1}{2}\zeta\langle\lambda_a K\rangle\langle\lambda_b K\rangle + \text{h.c.}\,. \tag{4.12}
$$

To get the weak functional one must replace in eq. (3.11) $\hat{\Gamma}_\mu$ by $\hat{\Gamma}^W_\mu$ and $\bar{\sigma}$ by $\hat{\sigma}^W$ once in each term and insert the appropriate binomial factors.

5. Mathematica© program Ampcalculator

The program Ampcalculator [18] evaluates the complete amplitudes for strong and non-leptonic weak mesonic processes to $O(p^4)$ and to $O(\phi^6)$ (counting an external photon as $O(\phi^2)$), with the following exceptions.

- In the Wess-Zumino-Witten functional [12] incorporating the chiral anomaly, only terms with explicit spin-1 fields are processed. In other words, we omit for practical reasons[2] the part that cannot be written as the integral of a finite polynomial in U, $\partial_\mu U$ in four dimensions but can be written as a five-dimensional integral, with Minkowski space as boundary of the five-dimensional manifold.

- For radiative semileptonic decays (e.g., for $K^+ \to l^+\nu_l\gamma$), the Bremsstrahlung amplitude(s) for the photon(s) coupling to the charged lepton must be added by hand. For instance, for a single photon in the final state the program only calculates hadronic matrix elements of the type

$$M_{\mu\nu} = i\int d^4x\, e^{iq.x}\langle \text{hadrons}|T\, V_\mu^{\text{elm}}(x)\,(V_\nu(0) - A_\nu(0))^{|\Delta S|=1}\,|\text{M(eson)}\rangle \qquad (5.1)$$

  and then multiplies the result with the appropriate factors to get the (partial) amplitude for the decay $M \to$ hadrons $+ l + \nu_l + \gamma$.

By default, the program reduces the one-loop amplitudes down to the basic loop functions A, B, C defined in appendix B. In certain cases, the result may be more transparent and even more suitable for numerical treatment without performing the recursion relations for the various loop integrals collected in appendix B. The user has the option to skip the recursion relations by setting the variable norecurs=1.

For running Ampcalculator, a list of external mesons, photons and $W^\pm$ must be specified together with their momenta in an input file (cf. appendix A). All particles are assumed to be incoming. The program checks for charge conservation and exits if the total charge of the listed particles is non-zero. For convenience and better readability of the output, the user can also provide two lists with scalar products of momenta and polarization vectors (also used for virtual photons for simplicity). The first list contains the scalar products themselves and the second one the corresponding scalar variables. In this way, momentum conservation can be implemented. The lists can also be used to redefine momenta (see section 5.1 for an example). For semileptonic transitions, the program calculates the $V - A$ hadronic amplitude and then multiplies with the appropriate factors to get the full amplitude (except for lepton Bremsstrahlung) including the lepton matrix elements l_μ or $\hat{l}_\mu$, where

$$\begin{aligned} l_\mu &= \bar{u}(p_{\nu_l})\gamma_\mu(1-\gamma_5)v(p_{l^+})\,, \\ \hat{l}_\mu &= \bar{u}(p_{l^-})\gamma_\mu(1-\gamma_5)v(p_{\bar{\nu}_l})\,. \end{aligned} \qquad (5.2)$$

[2]The associated vertices contain at least five meson fields.

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The masses of the pseudoscalar mesons (always in the isospin limit) need not be specified explicitly for the squares of meson momenta. However, the user should specify (for his/her own convenience) if the external photon(s) is (are) real: for an on-shell photon with momentum k and polarization vector $\varepsilon[k]$, the lists should contain the declarations for $k \cdot k = k \cdot \varepsilon[k] = 0$ (cf., e.g., section 5.3). In the amplitudes of $O(p^4)$, by default M_η^2 is replaced using the Gell-Mann-Okubo relation

$$M_\eta^2 = \frac{4}{3}(4M_K^2 - M_\pi^2)\,, \tag{5.3}$$

except as argument of a loop function where for better readability M_η^2 is always kept. The user has the option to keep M_η^2 everywhere. The complex conjugate of a quantity G (e.g., G_8, N_i, V_{us}, ...) is denoted generically as $\hat{G}$.

Running Ampcalculator may be quite time consuming, especially for nonleptonic weak processes, the time increasing strongly with the number of external states. However, the big advantage is that it is computer time rather than the physicist's time that is being consumed. Especially for exploratory purposes, the program provides options for calculating the tree amplitudes (at lowest order or to $O(p^4)$) and the loop amplitudes separately. Details can be found in appendix A where the input file for Ampcalculator is listed. The input file ampcalculator.nb and the subroutine ampcalculatorsub.nb are available for general use [18]. The results are written into a separate output file.

To perform CHPT calculations, another Mathematica© package, called PHI [19], exists. That package makes use of the more general package FeynCalc [20]. In contrast to Ampcalculator, FeynCalc/PHI is not a fully automatic program. In order to use FeynCalc/PHI, detailed knowledge of Mathematica© and of the packages FeynCalc and PHI is needed. Another difference between FeynCalc/PHI and Ampcalculator is the way the amplitudes are calculated. FeynCalc/PHI generates and uses Feynman rules whereas Ampcalculator performs a functional differentiation of the generating functional. Both packages are completely independent allowing for an excellent check of the results.

In the remainder of this section we present input and output for three examples. Although the examples are nontrivial from the point-of-view of computer time needed the amplitudes are relatively simple and can easily be compared with results available in the literature. The indicated execution times were obtained running Mathematica©5 on a PC with 512 MB RAM at 1.6 GHz.

In addition to the three examples shown below, we have checked the existing results for the processes $\gamma\gamma \to \pi^+\pi^-$ [21], $e^+e^- \to 4\pi$ [7, 22], $K^+\pi^+ \to K^+\pi^+$ [23] and $K^0 \to 3\pi^0$ [25, 24].

5.1 The semileptonic decay $K^+ \to \pi^0 l^+ \nu_l$ (K^+_{l3})

The program Ampcalculator evaluates the hadronic $V - A$ amplitude and then multiplies it with the appropriate factor to get the full decay amplitude. With the external momenta defined through

$$K^+(p) \to \pi^0(p') l^+(p_l) \nu_l(p_\nu)$$

with

$$q = p_l + p_\nu = p - p' \,, \qquad P = p + p' \,, \tag{5.4}$$

a possible input for the lists ExternalParticles, replist1 and replist2 is given below. Of course, the choice of momenta and kinematic variables is up to the user. Remember that the program interprets all particles as incoming.

ExternalParticles = {K_+[p], W_-[Q], π_0[r]};

replist1 = {sp[Q, Q], sp[r, Q], sp[p, Q], sp[p, r], p^μ, r^μ, Q^μ};

replist2 = {t, ($\mathsf{M}_K^2-\mathsf{M}_\pi^2-$t)/2, ($-\mathsf{M}_K^2+\mathsf{M}_\pi^2-$t)/2, ($-\mathsf{M}_K^2-\mathsf{M}_\pi^2+$t)/2, ($\mathsf{P}^\mu+\mathsf{q}^\mu$)/2, ($-\mathsf{P}^\mu+\mathsf{q}^\mu$)/2, $-\mathsf{q}^\mu$};

The complete result of $O(p^4)$ is printed out in a separate output file (execution time: 0.30 hours).

Complete amplitude to O(p^4)

$$
\begin{aligned}
G_F l_\mu \hat{V}_{us} \Bigg(& \frac{P^\mu}{2} + \frac{2q^\mu L_5^r \left({M_K}^2 - {M_\pi}^2\right)}{{F_\pi}^2} + \frac{L_9^r \left(P^\mu t + q^\mu \left(-{M_K}^2 + {M_\pi}^2\right)\right)}{{F_\pi}^2} + \\
& + \frac{\left(t - 5{M_K}^2 - {M_\pi}^2\right)\left(P^\mu t + q^\mu\left(-{M_K}^2 + {M_\pi}^2\right)\right)}{192\pi^2 t {F_\pi}^2} - \\
& - \frac{\left(-2q^\mu \left({M_K}^2 - {M_\pi}^2\right)\left(t - 4{M_K}^2 + 2{M_\pi}^2\right) + P^\mu t\left(t - 8{M_K}^2 + 4{M_\pi}^2\right)\right)\bar{A}[{M_K}^2]}{8t{F_\pi}^2\left({M_K}^2 - {M_\pi}^2\right)} - \\
& - \frac{\left(P^\mu t\left(t - 4{M_K}^2\right) + 4q^\mu\left(t + {M_K}^2\right)\left({M_K}^2 - {M_\pi}^2\right)\right)\bar{A}[{M_\pi}^2]}{16t{F_\pi}^2\left({M_K}^2 - {M_\pi}^2\right)} + \\
& + \frac{3\left(P^\mu t\left(t - 4{M_K}^2\right) + 4q^\mu {M_K}^2\left({M_K}^2 - {M_\pi}^2\right)\right)\bar{A}[{M_\eta}^2]}{16t{F_\pi}^2\left({M_K}^2 - {M_\pi}^2\right)} + \\
& + \bigg(P^\mu t \left({M_K}^4 + \left(3t + {M_\pi}^2\right)^2 - 2{M_K}^2\left(21t + {M_\pi}^2\right)\right) - \\
& \quad - 4q^\mu\left({M_K}^2 - {M_\pi}^2\right)\left({M_K}^4 + 3t{M_\pi}^2 + {M_\pi}^4 - {M_K}^2\left(9t + 2{M_\pi}^2\right)\right)\bigg) \times \\
& \times \frac{\bar{B}[t, {M_K}^2, {M_\eta}^2]}{144t^2{F_\pi}^2} + \\
& + \bigg(-4q^\mu\left({M_K}^2 - {M_\pi}^2\right)\left(-t^2 + {M_K}^4 - 2{M_K}^2{M_\pi}^2 + {M_\pi}^4\right) + \\
& \quad + P^\mu t\left({M_K}^4 + \left(t - {M_\pi}^2\right)^2 - 2{M_K}^2\left(t + {M_\pi}^2\right)\right)\bigg)\frac{\bar{B}[t, {M_\pi}^2, {M_K}^2]}{16t^2{F_\pi}^2}\Bigg)
\end{aligned}
$$

The result agrees with ref. [26].

5.2 The nonleptonic decay $K^- \to \pi^-\pi^0$

In this decay, the complex conjugate coupling constants $\hat{G}_{27}$ and $\hat{R}_i^r$ appear in the amplitude. The assignment of momenta and scalar products is straightforward.

ExternalParticles = $\{$K$_-$[p$_1$], π_+[q$_1$], π_0[q$_2$]$\}$;

replist1 = $\{$sp[p$_1$, q$_1$], sp[p$_1$, q$_2$], sp[q$_1$, q$_2$]$\}$;

replist2 = $\{-\frac{1}{2}\,\mathrm{M}_K^2, -\frac{1}{2}\,\mathrm{M}_K^2,\ -\mathrm{M}_\pi^2+\frac{1}{2}\,\mathrm{M}_K^2\}$;

Complete amplitude to O(pˆ4)

$$\begin{aligned}
i\hat{G}_{27}\Bigg(&\frac{5}{3}\,F_\pi\,\left({M_K}^2-{M_\pi}^2\right)-\frac{80\,L_4^r\,\left(2\,{M_K}^4-{M_K}^2\,{M_\pi}^2-{M_\pi}^4\right)}{3\,F_\pi}-\\
&-\frac{20\,L_5^r\,\left({M_K}^4+2\,{M_K}^2\,{M_\pi}^2-3\,{M_\pi}^4\right)}{3\,F_\pi}+\frac{5\,\left({M_K}^4-3\,{M_K}^2\,{M_\pi}^2+2\,{M_\pi}^4\right)}{96\,\pi^2\,F_\pi}+\\
&+\frac{5\,\left({M_K}^4+3\,{M_K}^2\,{M_\pi}^2-4\,{M_\pi}^4\right)\,\hat{R}_8^r}{3\,F_\pi}+\frac{5\,\left({M_K}^4-{M_K}^2\,{M_\pi}^2\right)\,\hat{R}_9^r}{3\,F_\pi}+\\
&+\frac{10\,\left({M_K}^2-{M_\pi}^2\right)\left(2\,{M_K}^2+{M_\pi}^2\right)\hat{R}_{10}^r}{3\,F_\pi}+\frac{20\,{M_\pi}^2\,\left({M_K}^2-{M_\pi}^2\right)\hat{R}_{12}^r}{3\,F_\pi}+\\
&+\frac{5\,\left(3\,{M_K}^2+{M_\pi}^2\right)\bar{A}[{M_K}^2]}{12\,F_\pi}-\frac{5\,\left(4\,{M_K}^4-22\,{M_K}^2\,{M_\pi}^2+29\,{M_\pi}^4\right)\bar{A}[{M_\pi}^2]}{24\,F_\pi\,{M_\pi}^2}+\\
&+\frac{5\,{M_\pi}^2\,\bar{A}[{M_\eta}^2]}{8\,F_\pi}-\frac{5\,\left({M_K}^4-3\,{M_K}^2\,{M_\pi}^2+2\,{M_\pi}^4\right)\,\bar{B}[{M_K}^2,{M_\pi}^2,{M_\pi}^2]}{6\,F_\pi}+\\
&+\frac{5\,{M_K}^4\,\left(-{M_K}^2+{M_\pi}^2\right)\,\bar{B}[{M_\pi}^2,{M_K}^2,{M_\eta}^2]}{72\,F_\pi\,{M_\pi}^2}-\\
&-\frac{5\,{M_K}^2\,\left(5\,{M_K}^4-13\,{M_K}^2\,{M_\pi}^2+8\,{M_\pi}^4\right)\,\bar{B}[{M_\pi}^2,{M_\pi}^2,{M_K}^2]}{24\,F_\pi\,{M_\pi}^2}\Bigg)
\end{aligned}$$

With the appropriate renaming of coupling constants, the result (execution time: 1.48 hours) agrees with previous calculations in ref. [25, 27].

5.3 The nonleptonic decay $K^0 \to \pi^0\gamma\gamma$

Finally, we present the amplitude for a decay where three-propagator loops contribute:

$$K^0(p) \to \pi^0(p')\gamma(q_1)\gamma(q_2)\,.$$

In the limit of CP conservation, the even-intrinsic-parity amplitude determines the amplitude for $K_L \to \pi^0\gamma\gamma$ whereas the part with the ϵ-tensor yields the corresponding K_S amplitude. The following assignments (all particles incoming) correspond to $p_1 = -p'$, $k_1 = -q_1$, $k_2 = -q_2$.

ExternalParticles = $\{$K$_0$[p], $\pi_0[p_1]$, $\gamma[k_1]$, $\gamma[k_2]\}$;

replist1 = {sp[p,p$_1$], sp[k$_1$,k$_2$], sp[k$_1$, ϵ[k$_1$]], sp[k$_1$, ϵ[k$_2$]], sp[k$_2$, ϵ[k$_1$]], sp[k$_2$, ϵ[k$_2$]], sp[ϵ[k$_1$], ϵ[k$_2$]], sp[p,ϵ[k$_1$]], sp[p,ϵ[k$_2$]], sp[p$_1$, ϵ[k$_1$]], sp[p$_1$, ϵ[k$_2$]], sp[k$_1$,k$_1$], sp[k$_2$,k$_2$]};

replist2 = {$\frac{1}{2}$ (s-M$_K^2$-M$_\pi^2$), $\frac{s}{2}$, 0, -q$_1\cdot\epsilon_2$, -q$_2\cdot\epsilon_1$, 0, $\epsilon_1\cdot\epsilon_2$, p$\cdot\epsilon_1$, p$\cdot\epsilon_2$, -p$\cdot\epsilon_1$+q$_2\cdot\epsilon_1$, -p$\cdot\epsilon_2$+q$_1\cdot\epsilon_2$, 0, 0};

Complete amplitude to O(p^4)

$$
\begin{aligned}
&\frac{1}{2}e^2\Bigg(G_8\Bigg(\frac{(2\,q_1\cdot\epsilon_2\,q_2\cdot\epsilon_1-s\,\epsilon_1\cdot\epsilon_2)\,M_K{}^2}{4\sqrt{2}\,\pi^2\,s}+\\
&\qquad+\frac{4}{s}\sqrt{2}C[0,0,-\frac{s}{2},M_\pi{}^2,M_\pi{}^2,M_\pi{}^2]\,(2\,q_1\cdot\epsilon_2 q_2\cdot\epsilon_1-s\epsilon_1\cdot\epsilon_2)\,M_\pi{}^2\left(s-M_\pi{}^2\right)+\\
&\qquad+\frac{4}{s\left(M_K{}^2-M_\pi{}^2\right)}\sqrt{2}C[0,0,-\frac{s}{2},M_K{}^2,M_K{}^2,M_K{}^2]\times\\
&\qquad\times(-2q_1\cdot\epsilon_2 q_2\cdot\epsilon_1+s\epsilon_1\cdot\epsilon_2)\,M_K{}^2\left(s\,M_K{}^2-M_K{}^4-s\,M_\pi{}^2+M_\pi{}^4\right)\Bigg)+\\
&\quad+G_{27}\Bigg(\frac{(2\,q_1\cdot\epsilon_2\,q_2\cdot\epsilon_1-s\,\epsilon_1\cdot\epsilon_2)\left(-13\,M_K{}^2+15\,\left(s-M_\pi{}^2\right)\right)}{12\sqrt{2}\,\pi^2\,s}+\\
&\qquad+\frac{1}{3\,s\,\left(M_K{}^2-M_\pi{}^2\right)}2\sqrt{2}\,C[0,0,-\frac{s}{2},M_\pi{}^2,M_\pi{}^2,M_\pi{}^2]\times\\
&\qquad\times(2\,q_1\cdot\epsilon_2\,q_2\cdot\epsilon_1-s\,\epsilon_1\cdot\epsilon_2)\,M_\pi{}^2\times\\
&\qquad\times\left(-5\,M_K{}^4+24\,M_\pi{}^2\,\left(-s+M_\pi{}^2\right)+M_K{}^2\,\left(9\,s+M_\pi{}^2\right)\right)+\\
&\qquad+\frac{1}{3\,s\,\left(M_K{}^2-M_\pi{}^2\right)}2\sqrt{2}\,C[0,0,-\frac{s}{2},M_K{}^2,M_K{}^2,M_K{}^2]\times\\
&\qquad\times(-2\,q_1\cdot\epsilon_2 q_2\cdot\epsilon_1+s\epsilon_1\cdot\epsilon_2)\times\\
&\qquad\times M_K{}^2\left(21\,M_K{}^4+6\,M_\pi{}^2\,\left(s-M_\pi{}^2\right)+M_K{}^2\,\left(-21\,s+5\,M_\pi{}^2\right)\right)\Bigg)\Bigg)+\\
&\quad+\frac{1}{2}\,i\,e^2\Bigg(\frac{2\sqrt{2}\epsilon^{\xi\rho\sigma\tau}G_8\left(s-M_K{}^2\right)\left(M_K{}^2-M_\pi{}^2\right)\left(p_\xi k_{1\rho}-k_{1\xi}p_{1\rho}\right)(\epsilon(k_1))_\sigma(\epsilon(k_2))_\tau}{\pi^2\left(s-M_\pi{}^2\right)\left(3\,s-4\,M_K{}^2+M_\pi{}^2\right)}-\\
&\qquad-\frac{2\sqrt{2}\epsilon^{\xi\rho\sigma\tau}G_{27}\left(s-M_K{}^2\right)\left(M_K{}^2-M_\pi{}^2\right)\left(p_\xi k_{1\rho}-k_{1\xi}p_{1\rho}\right)(\epsilon(k_1))_\sigma(\epsilon(k_2))_\tau}{\pi^2\left(s-M_\pi{}^2\right)\left(3s-4M_K{}^2+M_\pi{}^2\right)}\Bigg)
\end{aligned}
$$

The amplitude (execution time: 1.68 hours) agrees with what is available in the literature [28, 29]. The function F defined in ref. [28] is related to the three-propagator loop function as

$$F(s/M^2)=1+32\pi^2C\left(0,0,\frac{-s}{2},M^2,M^2,M^2\right). \tag{5.5}$$

In the anomaly contribution, M_η^2 was replaced via the Gell-Mann-Okubo relation. To our knowledge (and for good reasons), the 27-plet kaon loop amplitude and the anomalous contribution proportional to G_{27} have not been published before.

6. Conclusions

The generating functional is a convenient quantity for a compact representation of Green functions and S-matrix elements. Especially for an effective field theory like chiral perturbation theory, with its many coupling constants and derivative couplings, the standard diagrammatic calculations can be quite cumbersome and they must be performed for each process separately. The great advantage of the generating functional is that the renormalized amplitudes can be obtained once and for all.

Extending previous work, we have presented in this paper the generating functional of Green functions for chiral SU(3) for both strong and nonleptonic weak interactions in the meson sector to next-to-leading order in the low-energy expansion and for at most six external states, a photon counting as $O(\phi^2)$. Such processes require the inclusion of one-loop diagrams with up to three meson propagators and they comprise almost all strong and nonleptonic weak mesonic transitions of physical interest.

The main purpose of this work is to reproduce and to check previous calculations in a simple and straightforward way but the functional presented here has also been used to derive new results [10]. Although the representation is compact the actual extraction of a specific amplitude via three-dimensional flavour traces can be quite laborious, the toil increasing of course with the number of external states and especially for nonleptonic weak transitions. We have therefore provided as an essential part of this work the Mathematica© program Ampcalculator for general use that performs the necessary manipulations upon input of the relevant external particles with their momenta.

We have described the necessary input for the calculation of three semileptonic and nonleptonic K decay amplitudes and presented the resulting output. As far as available, the results agree with previous calculations.

The generating functional presented here can be extended in several ways. In addition to radiative corrections for strong processes [8, 9], photons and leptons can be included as dynamical degrees of freedom for similar treatments of radiative corrections for semileptonic [30] and nonleptonic weak decays.

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A. Input for the program Ampcalculator

We list here the contents of the input file ampcalculator.nb. During execution of the input file, the subroutine ampcalculatorsub.nb is called where the actual computations are performed. The result is then printed out in a separate output file.

Program Ampcalculator

The program generates the amplitude for any strong or nonleptonic weak process of pseudoscalar mesons, external photons and W bosons to next-to-leading order in the low-energy expansion (chiral SU(3)) with at most six external particles (a photon counts as two particles), at most one W (semileptonic decays) and with at most three propagators in the loop. The W is assumed to couple to a lepton pair indicated in the output by the leptonic matrix elements l_μ and $\hat{l}_\mu$, defined by

$$l_\mu = \bar{u}(p_\nu)\gamma_\mu(1-\gamma_5)v(p_{l^+})\,, \qquad \hat{l}_\mu = \bar{u}(p_{l^-})\gamma_\mu(1-\gamma_5)v(p_\nu)\,,$$

where p_ν, p_l are the momenta of the neutrino and of the charged lepton. For radiative semileptonic decays, Bremsstrahlung off the charged lepton is not included and must be added by hand. In the anomaly functional, only terms containing external gauge fields are included.

Select up to six particles (a photon counts as two particles) of the list $\{\pi_+,\ \pi_-,\ \pi_0,\ K_+,\ K_-,\ K_0,\ \bar{K}_0,\ \eta_8,\ \gamma,\ W_+,\ W_-\}$. Assign a momentum to each particle. In the list *ExternalParticles* all particles are supposed to be incoming. Define the scalar products sp[p_i, p_j] that you want to replace by polynomials of kinematic variables in the lists *replist1* and *replist2*. One should not use the capital letters F, G, L, N and R in the lists *replist1*, *replist2* and *ExternalParticles* as they are reserved for coupling constants. Small l is reserved for the leptonic matrix element. For semileptonic decays the index μ should only be used for momenta that are contracted with the leptonic matrix elements $\mathtt{l}_\mu$ or $\hat{\mathtt{l}}_\mu$. Isospin conservation is assumed.

The program terminates if either

(i) some of the external particles are not contained in the list above,

(ii) charge is not conserved,

(iii) there are more than one external W,

(iv) strangeness change $|\Delta\mathtt{S}| > 1$.

Notation:

$\bar{A}[M^2]$ is the renormalized single-propagator loop integral $\bar{A}[M^2] = -\dfrac{M^2}{(4\pi)^2}\,log(M^2/\mu^2)$, where M is a meson mass and μ is the renormalization scale. $\bar{B}[s, M_1^2, M_2^2]$ is the standard two-propagator loop function subtracted at s=0, i.e. $\bar{B}[0, M_1^2, M_2^2]$=0. All loop functions and recursion relations can be found in appendix B. G_8, G_{27} are the nonleptonic weak LECs of order p^2. L_i^r, N_i^r, R_i^r are the strong, weak octet and 27-plet LECs of O(p^4), respectively, all renormalized at scale μ. The corresponding lagrangians are reproduced in appendix C. $\hat{G}_8, \hat{G}_{27}, \hat{N}_i^r, \hat{R}_i^r$, are complex conjugate LECs.

Options:

Set the following variables equal to 1 to get partial results (otherwise the variables should be set to 0):

onlytreep2=1 amplitude of O(p^2)
onlytreep2and4=1 tree level amplitude up to O(p^4)
onlyloops=1 one-loop amplitude only

By default (nogmo=0), the Gell-Mann-Okubo relation is applied in the amplitudes of O(p^4) to express M_η^2 in terms of M_K^2, M_π^2. For better readability, M_η^2 is never replaced as argument in loop functions. Setting nogmo=1, M_η^2 is kept everywhere. If the recursion relations for the loop integrals should not be performed set norecurs=1.

onlytreep2 = 0;
onlytreep2and4 = 0;
onlyloops = 0;
nogmo = 0;
norecurs=0;

*** Example: ***
ExternalParticles = $\{\pi_0[p_1],\pi_0[p_2],\pi_+[p_3],\pi_-[p_4]\}$;
(* scattering amplitude for $\pi_0[p_1]$ + $\pi_0[p_2] \rightarrow\ \pi_-[-p_3]$ + $\pi_+[-p_4]$ *)
replist1 =$\{$sp[p_1,p_2],sp[p_1,p_3],sp[p_1,p_4], sp[p_2,p_3],sp[p_2,p_4],sp[p_3,p_4]$\}$;
replist2 =$\{\frac{1}{2}\left(s-2M_\pi^2\right), \frac{1}{2}\left(t-2M_\pi^2\right), \frac{1}{2}\left(u-2M_\pi^2\right), \frac{1}{2}\left(u-2M_\pi^2\right), \frac{1}{2}\left(t-2M_\pi^2\right), \frac{1}{2}\left(s-2M_\pi^2\right)\}$;
(* *)
(* Insert the correct path in the next line.*)

```
notesub=NotebookOpen["C:\Math5\programs\Unterdorfer\ampcalculatorsub.nb"];

SelectionMove[notesub,All,Notebook]

SelectionEvaluate[Notebook]
```

B. Loop integrals and constituent functions

The standard functions A, B, B_{ij}, C and C_{ij} occurring in one-loop integrals with up to three propagators are defined through the following relations (C_F denotes the Feynman integration contour):

$$A \doteq \frac{1}{i}\int_{C_F}\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2-M_P^2)}\,, \tag{B.1}$$

$$\{X\} \doteq \frac{1}{i}\int_{C_F}\frac{d^dk}{(2\pi)^d}\,\frac{X}{(k^2-M_P^2)((k-p)^2-M_Q^2)}\,, \tag{B.2}$$

$$B(p^2) \doteq \{1\}\,, \tag{B.3}$$

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$$\{k_\mu\} = p_\mu B_{11}(p^2)\,, \tag{B.4}$$

$$\{k_\mu k_\nu\} = g_{\mu\nu} B_{20}(p^2) + p_\mu p_\nu B_{22}(p^2)\,, \tag{B.5}$$

$$\{\{X\}\} \doteq \frac{1}{i}\int_{C_F} \frac{d^d k}{(2\pi)^d} \frac{X}{(k^2 - M_P^2)((k-p_a)^2 - M_Q^2)((k-p_b)^2 - M_R^2)}\,, \tag{B.6}$$

$$C \doteq \{\{1\}\}\,, \tag{B.7}$$

$$\{\{k_\mu\}\} = p_{a\mu}\, C_{11a} + p_{b\mu}\, C_{11b}\,, \tag{B.8}$$

$$\{\{k_\mu k_\nu\}\} = g_{\mu\nu}\, C_{20} + p_{a\mu}p_{a\nu}\, C_{22a} + p_{b\mu}p_{b\nu}\, C_{22b} + (p_{a\mu}p_{b\nu} + p_{b\mu}p_{a\nu})\tilde{C}_{22}\,, \tag{B.9}$$

$$\begin{aligned} \{\{k_\mu k_\nu k_\lambda\}\} = {} & p_{a\mu}p_{a\nu}p_{a\lambda}\, C_{33a} + p_{b\mu}p_{b\nu}p_{b\lambda} C_{33b} + \\ & +(p_{a\mu}p_{a\nu}p_{b\lambda} + p_{a\mu}p_{b\nu}p_{a\lambda} + p_{b\mu}p_{a\nu}p_{a\lambda})\tilde{C}_{33a} + \\ & +(p_{b\mu}p_{b\nu}p_{a\lambda} + p_{b\mu}p_{a\nu}p_{b\lambda} + p_{a\mu}p_{b\nu}p_{b\lambda})\tilde{C}_{33b} + \\ & +(p_{a\mu}g_{\nu\lambda} + p_{a\nu}g_{\mu\lambda} + p_{a\lambda}g_{\mu\nu})C_{31a} + (p_{b\mu}g_{\nu\lambda} + p_{b\nu}g_{\mu\lambda} + p_{b\lambda}g_{\mu\nu})C_{31b}\,, \end{aligned} \tag{B.10}$$

with $C_{ij\,a} = C_{ij}(p_a^2, p_b^2, p_a\cdot p_b, M_P^2, M_Q^2, M_R^2)$, $C_{ij\,b} = C_{ij}(p_b^2, p_a^2, p_a\cdot p_b, M_P^2, M_R^2, M_Q^2)$ and $C_{ij} = C_{ij\,a} = C_{ij\,b}$. The functions A and B can be split into a finite and a divergent part for $d \to 4$:

$$A(M_P^2) = \bar{A}(M_P^2) - 2M_P^2\,\Lambda(\mu)\,, \qquad B(p^2) = \bar{B}(p^2) + B(0)\,, \tag{B.11}$$

where

$$\bar{A}(M_P^2) = -\frac{M_P^2}{(4\pi)^2}\ln\frac{M_P^2}{\mu^2}\,, \qquad \Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2}\Big(\frac{1}{d-4} - \frac{1}{2}(\ln(4\pi) + \Gamma'(1) + 1)\Big)\,, \tag{B.12}$$

$$\begin{aligned} \bar{B}(p^2) = {} & \frac{1}{32\pi^2}\left(2 + \frac{M_P^2 - M_Q^2}{p^2}\ln\frac{M_Q^2}{M_P^2} - \frac{M_P^2 + M_Q^2}{M_P^2 - M_Q^2}\ln\frac{M_Q^2}{M_P^2} - \frac{\sqrt{\lambda(p^2, M_P^2, M_Q^2)}}{p^2}\,\times\right. \\ & \left.\times \ln\frac{(p^2 + \sqrt{\lambda(p^2, M_P^2, M_Q^2)})^2 - (M_P^2 - M_Q^2)^2}{(p^2 - \sqrt{\lambda(p^2, M_P^2, M_Q^2)})^2 - (M_P^2 - M_Q^2)^2}\right), \\ \lambda(x,y,z) = {} & x^2 + y^2 + z^2 - 2(xy + yz + zx)\,, \end{aligned} \tag{B.13}$$

and

$$B(0) = -2\Lambda(\mu) + \frac{\bar{A}(M_P^2) - \bar{A}(M_Q^2)}{M_P^2 - M_Q^2}\,. \tag{B.14}$$

The coefficient functions are of the form

$$\begin{aligned} B_{11}(p^2) &= \frac{-A(M_P{}^2) + A(M_Q{}^2) + B(p^2)\left(M_P{}^2 - M_Q{}^2 + p^2\right)}{2p^2}\,, \\ B_{20}(p^2) &= -\frac{p^2 - 3M_P{}^2 - 3M_Q{}^2}{288\pi^2} + \frac{A(M_Q{}^2) + 2B(p^2)M_P{}^2 - \left(M_P{}^2 - M_Q{}^2 + p^2\right)B_{11}(p^2)}{6}\,, \\ B_{22}(p^2) &= \frac{p^2 - 3M_P{}^2 - 3M_Q{}^2}{288\pi^2 p^2} + \frac{A(M_Q{}^2) - B(p^2)M_P{}^2 + 2\left(M_P{}^2 - M_Q{}^2 + p^2\right)B_{11}(p^2)}{3p^2}\,. \end{aligned} \tag{B.15}$$

The pure C function is given by

$$C(p_a^2, p_b^2, p_a\cdot p_b, M_P^2, M_Q^2, M_R^2) = -\frac{1}{16\pi^2}\frac{1}{\sqrt{\lambda}}\sum_{i=1}^{3}\sum_{\sigma=\pm}\left(\mathrm{Li}_2\left(\frac{x_i}{x_i - z_i^\sigma}\right) - \mathrm{Li}_2\left(\frac{x_i - 1}{x_i - z_i^\sigma}\right)\right) \tag{B.16}$$

with the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t} \ln(1-t) \tag{B.17}$$

and

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(1 + \frac{p_a^2 - p_b^2 - p_D^2}{\sqrt{\lambda}}\right) + \\
&\quad + \frac{1}{2p_a^2}\left(- M_P^2\left(1 + \frac{p_a^2 + p_D^2 - p_b^2}{\sqrt{\lambda}}\right) + M_Q^2\left(1 - \frac{p_a^2 - p_D^2 + p_b^2}{\sqrt{\lambda}}\right)\right) + \frac{M_R^2}{\sqrt{\lambda}}\,, \\
z_1^{\pm} &= \frac{1}{2}\left(1 - \frac{M_P^2 - M_Q^2}{p_a^2} \pm \sqrt{\left(1 - \frac{M_P^2 - M_Q^2}{p_a^2}\right)^2 - \frac{4}{p_a^2}(M_Q^2 - \mathrm{i}\,\epsilon)}\right)\,, \\
p_D^2 &= p_a^2 - 2\,p_a \cdot p_b + p_b^2\,, \qquad \lambda = \lambda(p_a^2, p_b^2, p_D^2)\,.
\end{aligned} \tag{B.18}$$

The quantities x_2, z_2^σ and x_3, z_3^σ are obtained by cyclic interchanges of (p_a^2, p_D^2, p_b^2) and (M_P^2, M_Q^2, M_R^2).

In the following $B(1,3)$ is defined as $B(p_b^2, M_P^2, M_R^2)$ and $B(2,3)$ as $B(p_a^2 - 2\,p_a \cdot p_b + p_b^2, M_Q^2, M_R^2)$. The same notation is valid for B_{11}, B_{20} and B_{22}.

$$\begin{aligned}
f_1 &= p_a^2 + M_P^2 - M_Q^2\,, \\
C_{11\,a} &= \frac{H_{11\,a}\, {p_b}^2 - H_{11\,b}\, p_a \cdot p_b}{{p_a}^2\, {p_b}^2 - p_a \cdot {p_b}^2}\,, \\
C_{20} &= \frac{1}{64\,\pi^2} + \frac{2\,C\,M_P^2 + B(2,3) - C_{11\,a}\, f_1 - C_{11\,b}\,({M_P}^2 - {M_R}^2 + {p_b}^2)}{4}\,, \\
C_{22\,a} &= \frac{H_{21\,a}\, {p_b}^2 - H_{22\,b}\, p_a \cdot p_b}{{p_a}^2\, {p_b}^2 - p_a \cdot {p_b}^2}\,, \qquad \tilde{C}_{22} = \frac{H_{22\,b} {p_a}^2 - H_{21\,a}\, p_a \cdot p_b}{{p_a}^2\, {p_b}^2 - p_a \cdot {p_b}^2}\,, \\
C_{33\,a} &= \frac{H_{31\,a} {p_b}^2 - H_{32\,b} p_a \cdot p_b}{{p_a}^2 {p_b}^2 - p_a \cdot {p_b}^2}\,, \qquad \tilde{C}_{33\,a} = \frac{H_{32\,b}\, {p_a}^2 - H_{31\,a} p_a \cdot p_b}{{p_a}^2\, {p_b}^2 - p_a \cdot {p_b}^2}\,, \\
C_{31\,a} &= \frac{H_{30\,a}\, {p_b}^2 - H_{30\,b} p_a \cdot p_b}{{p_a}^2\, {p_b}^2 - p_a \cdot {p_b}^2}\,, \\
H_{11\,a} &= \frac{-B(1,3) + B(2,3) + C\, f_1}{2}\,, \\
H_{21\,a} &= \frac{B(2,3) + C_{11\,a}\, f_1 - B_{11}(2,3)}{2} - C_{20}\,, \\
H_{22\,a} &= \frac{C_{11\,b}\, f_1 - B_{11}(1,3) + B_{11}(2,3)}{2}\,, \\
H_{30\,a} &= \frac{C_{20}\, f_1 - B_{20}(1,3) + B_{20}(2,3)}{2}\,, \\
H_{31\,a} &= \frac{B(2,3) + f_1\, C_{22\,a} - 2\,B_{11}(2,3) + B_{22}(2,3)}{2} - 2C_{31a}\,, \\
H_{32\,a} &= \frac{f_1 C_{22\,b} - B_{22}(1,3) + B_{22}(2,3)}{2}\,.
\end{aligned} \tag{B.19}$$

The results for the integrals agree with those in ref. [31].

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The constituent functions of the one-loop functional are of the following form:

$$
\begin{aligned}
G^{\mu\nu}_{1\,PQ}(p) &= p^\mu p^\nu\, a(p^2, M_P^2, M_Q^2) + g^{\mu\nu}\, b(p^2, M_P^2, M_Q^2)\,,\\
G^{\mu}_{2\,PQ}(p) &= i\, p^\mu \left(\frac{1}{2}\, B(p^2, M_P^2, M_Q^2) - B_{11}(p^2, M_P^2, M_Q^2)\right),\\
G_{3\,PQ}(p) &= \frac{1}{4} B(p^2, M_P^2, M_Q^2)\,,\\
G^{\mu\nu\lambda}_{4\,PQR}(p_a, p_b) &= i\,\big(c_a\, p_{a_\mu} p_{a_\nu} p_{a_\lambda} + c_b\, p_{b_\mu} p_{b_\nu} p_{b_\lambda} + d_a\, p_{a_\mu} p_{a_\nu} p_{b_\lambda} + e_a\, p_{a_\mu} p_{b_\nu} p_{a_\lambda} +\\
&\quad + f_a\, p_{b_\mu} p_{a_\nu} p_{a_\lambda} + d_b\, p_{b_\mu} p_{b_\nu} p_{a_\lambda} + f_b\, p_{b_\mu} p_{a_\nu} p_{b_\lambda} +\\
&\quad + e_b\, p_{a_\mu} p_{b_\nu} p_{b_\lambda} + g_a\, p_{a_\mu} g^{\nu\lambda} + h_a\, p_{a_\nu} g^{\mu\lambda} + i_a\, p_{a_\lambda} g^{\mu\nu} +\\
&\quad + h_b\, p_{b_\mu} g^{\nu\lambda} + g_b\, p_{b_\nu} g^{\mu\lambda} + i_b p_{b_\lambda} g^{\mu\nu}\big)\,,\\
G^{\mu\nu}_{5\,PQR}(p_a, p_b) &= j_a\, g^{\mu\nu} + k_a\, p_{a_\mu} p_{a_\nu} + k_b\, p_{b_\mu} p_{b_\nu} + l_a\, p_{a_\mu} p_{b_\nu} + m_a\, p_{b_\mu} p_{a_\nu}\,,\\
G^{\mu}_{6\,PQR}(p_a, p_b) &= i\,\big(n_a\, p_{a_\mu} + n_b\, p_{b_\mu}\big)\,, \qquad G_{7\,PQR}(p_a, p_b) = \frac{1}{6}\, C\,,
\end{aligned}
\tag{B.20}
$$

with for example

$$
c_a = c(p_a^2, p_b^2, p_a\cdot p_b, M_P^2, M_Q^2, M_R^2) \qquad \text{and} \qquad c_b = c(p_b^2, p_a^2, p_a\cdot p_b, M_P^2, M_R^2, M_Q^2)\,, \tag{B.21}
$$

where

$$
\begin{aligned}
a(p^2, M_P^2, M_Q^2) &= \frac{4(B_{11}(p^2, M_P^2, M_Q^2) - B_{22}(p^2, M_P^2, M_Q^2)) - B(p^2, M_P^2, M_Q^2)}{4}\,,\\
b(p^2, M_P^2, M_Q^2) &= -B_{20}(p^2, M_P^2, M_Q^2)\,,\\
c_a &= \frac{C_{11\,a} - 4\, C_{22\,a} + 4\, C_{33\,a}}{3}\,,\\
d_a &= \frac{C_{11\,a} - 2\, C_{22\,a} - 2\, \tilde{C}_{22} + 4\, \tilde{C}_{33\,a}}{3}\,,\\
e_a &= \frac{-C/2 + 2\, C_{11\,a} + C_{11\,b} - 2\, C_{22\,a} - 4\, \tilde{C}_{22} + 4\, \tilde{C}_{33\,a}}{3}\,,\\
f_a &= \frac{-2\left(\tilde{C}_{22} - 2\, \tilde{C}_{33\,a}\right)}{3}\,,\\
g_a &= \frac{-2\,(C_{20} - 2\, C_{31\,a})}{3}\,,\\
h_a &= \frac{4\, C_{31\,a}}{3}\,,\\
i_a &= \frac{-2\,(C_{20} - 2\, C_{31\,a})}{3}\,,\\
j_a &= -2\, C_{20}\,,\\
k_a &= C_{11\,a} - 2\, C_{22\,a}\,,\\
l_a &= -\frac{C}{2} + C_{11\,a} + C_{11\,b} - 2\, \tilde{C}_{22}\,,\\
m_a &= -2\, \tilde{C}_{22}\,,\\
n_a &= \frac{C}{2} - 2\, C_{11\,a}\,.
\end{aligned}
\tag{B.22}
$$

C. Lagrangians of $\mathbf{O(p^4)}$

The strong lagrangian of $O(p^4)$ is given by [3]

$$\begin{aligned}
\mathcal{L}_4^S =\ & L_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle + \\
& + L_3\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle + \\
& + L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)\rangle + L_6\langle \chi^\dagger U + \chi U^\dagger\rangle^2 + \\
& + L_7\langle \chi^\dagger U - \chi U^\dagger\rangle^2 + L_8\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle - \\
& - iL_9\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned} \tag{C.1}$$

with

$$F^R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - \mathrm{i}\,[r_\mu, r_\nu]\,, \qquad F^L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - \mathrm{i}\,[l_\mu, l_\nu]\,. \tag{C.2}$$

The nonleptonic weak lagrangian of $O(G_F p^4)$ consists of an octet and a 27-plet part:

$$\mathcal{L}_4^W = \mathcal{L}_4^{W,8} + \mathcal{L}_4^{W,27}\,. \tag{C.3}$$

Neglecting terms that would only contribute at $O(G_F^2)$ (with external $W^\pm$ fields in addition to the nonleptonic weak transition), the $\Delta S = 1$ octet lagrangian can be written as [6, 32]

$$\begin{aligned}
\mathcal{L}_4^{W,8} =\ & G_8 F^2 \{N_1\ \langle \lambda D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + N_2\ \langle \lambda D_\mu U^\dagger D^\nu U D_\nu U^\dagger D^\mu U\rangle + \\
& + N_3\ \langle \lambda D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle + N_4\ \langle \lambda D_\mu U^\dagger U\rangle\langle U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + \\
& + N_5\ \langle \lambda \{S, D_\mu U^\dagger D^\mu U\}\rangle + N_6\ \langle \lambda D_\mu U^\dagger U\rangle\langle U^\dagger D^\mu U S\rangle + \\
& + N_7\ \langle \lambda S\rangle\langle D_\mu U^\dagger D^\mu U\rangle + N_8\ \langle \lambda D_\mu U^\dagger D^\mu U\rangle\langle S\rangle + N_9\ i\langle \lambda [P, D_\mu U^\dagger D^\mu U]\rangle + \\
& + N_{10}\ \langle \lambda S^2\rangle + N_{11}\ \langle \lambda S\rangle\langle S\rangle - N_{12}\ \langle \lambda P^2\rangle - N_{13}\ \langle \lambda P\rangle\langle P\rangle + \\
& + N_{14}\ i\langle \lambda\{V^{\mu\nu}, D_\mu U^\dagger D_\nu U\}\rangle + N_{15}\ i\langle \lambda D_\mu U^\dagger U V^{\mu\nu} U^\dagger D_\nu U\rangle - \\
& - N_{16}\ i\langle \lambda\{A^{\mu\nu}, D_\mu U^\dagger D_\nu U\}\rangle - N_{17}\ i\langle \lambda D_\mu U^\dagger U A^{\mu\nu} U^\dagger D_\nu U\rangle + \\
& + 2N_{18}\ \langle \lambda (F_L^{\mu\nu} U^\dagger F_{R\mu\nu} U + U^\dagger F_{R\mu\nu} U F_L^{\mu\nu})\rangle + \\
& + N_{28}\ i\varepsilon_{\mu\nu\rho\sigma}\langle \lambda D^\mu U^\dagger U\rangle\langle U^\dagger D^\nu U D^\rho U^\dagger D^\sigma U\rangle + \\
& + 2N_{29}\ \varepsilon_{\mu\nu\rho\sigma}\langle \lambda [U^\dagger F_R^{\mu\nu} U, D^\rho U^\dagger D^\sigma U]\rangle + N_{30}\ \varepsilon_{\mu\nu\rho\sigma}\langle \lambda U^\dagger D^\mu U\rangle\langle V^{\rho\sigma} D^\nu U^\dagger U\rangle - \\
& - N_{31}\ \varepsilon_{\mu\nu\rho\sigma}\langle \lambda U^\dagger D^\mu U\rangle\langle A^{\rho\sigma} D^\nu U^\dagger U\rangle\} + \text{h.c.}
\end{aligned} \tag{C.4}$$

with

$$\begin{aligned}
S &= \chi^\dagger U + U^\dagger \chi\,, & P &= \mathrm{i}(\chi^\dagger U - U^\dagger \chi)\,, \\
V_{\mu\nu} &= U^\dagger F^R_{\mu\nu} U + F^L_{\mu\nu}\,, & A_{\mu\nu} &= U^\dagger F^R_{\mu\nu} U - F^L_{\mu\nu}\,.
\end{aligned} \tag{C.5}$$

For the 27-plet lagrangian we use the form given in ref. [33]:

$$\begin{aligned}
\mathcal{L}_4^{W,27} &= G_{27} F^2 \tilde{t}_{ij,kl} I_{ji,lk} + \text{h.c.}\,, \qquad\qquad (\text{C.6}) \\
I_{ij,kl} &= R_1\,(L^\mu L_\mu)_{ij}(L^\nu L_\nu)_{kl} + R_2\,(L_\mu L_\nu)_{ij}(L^\mu L^\nu)_{kl} + R_3\,(L_\mu L_\nu)_{ij}(L^\nu L^\mu)_{kl} + \\
& + R_4\,(L_\mu)_{ij}\,(L_\nu L^\mu L^\nu)_{kl} + R_5\,(L_\mu)_{ij}\,\{L^\mu,\, L_\nu L^\nu\}_{kl} + R_6\langle L^\mu L_\mu\rangle\,(L_\nu)_{ij}\ (L^\nu)_{kl} + \\
& + R_7 S_{ij}\,(L^\mu L_\mu)_{kl} + R_8\{S,\, L_\mu\}_{ij}(L^\mu)_{kl} + R_9\mathrm{i}[P,\, L_\mu]_{ij}(L^\mu)_{kl} +
\end{aligned}$$

$$
\begin{aligned}
&+R_{10}\,\langle S\rangle\,(L_\mu)_{ij}\,(L^\mu)_{kl}+R_{11}S_{ij}S_{kl}+R_{12}P_{ij}\,P_{kl}+R_{13}\mathrm{i}\,(v_{\mu\nu})_{ij}\,(L^\mu L^\nu)_{kl}+\\
&+R_{14}\mathrm{i}\,(a_{\mu\nu})_{ij}\,(L^\mu L^\nu)_{kl}+R_{15}\mathrm{i}\,[v_{\mu\nu},L^\mu]_{ij}\,(L^\nu)_{kl}+R_{16}\mathrm{i}\,[a_{\mu\nu},\,L^\mu]_{ij}\,(L^\nu)_{kl}+\\
&+R_{17}\,(v_{\mu\nu})_{ij}\,(v^{\mu\nu})_{kl}+R_{18}\,(v_{\mu\nu})_{ij}\,(a^{\mu\nu})_{kl}+R_{19}\,(a_{\mu\nu})_{ij}\,(a^{\mu\nu})_{kl}+\\
&+R_{20}\mathrm{i}\epsilon^{\mu\nu\rho\sigma}(L_\mu\,L_\nu)_{ij}\,(L_\rho\,L_\sigma)_{kl}+R_{21}\mathrm{i}\epsilon^{\mu\nu\rho\sigma}(L_\mu)_{ij}(L_\nu L_\rho L_\sigma)_{kl}+\\
&+R_{22}\,\epsilon^{\mu\nu\rho\sigma}(L_\mu)_{ij}(L_\nu v_{\rho\sigma})_{kl}+R_{23}\,\epsilon^{\mu\nu\rho\sigma}(L_\mu)_{ij}(L_\nu\,a_{\rho\sigma})_{kl}\,. \qquad \text{(C.7)}
\end{aligned}
$$

The coefficients $\tilde{t}_{ij,kl}$ are defined as

$$
\begin{aligned}
\tilde{t}_{12,31}&=\tilde{t}_{31,12}=\tilde{t}_{32,11}=\tilde{t}_{11,32}=\frac{1}{3}\,,\\
\tilde{t}_{22,32}&=\tilde{t}_{32,22}=\tilde{t}_{32,33}=\tilde{t}_{33,32}=-\frac{1}{6}\,,\\
\tilde{t}_{ij,kl}&=0 \qquad \text{otherwise}\,. \qquad \text{(C.8)}
\end{aligned}
$$

The low-energy constants L_i, N_i and R_i are dimensionless. Moreover, the L_i are real whereas G_8, G_{27}, N_i, R_i can be complex in the presence of CP violation.

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